Lost in Mathematics: Quantum Field Theory

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1 Harmonic oscillators: quantization of vacuum

Following the questionable "quantization" of Gordon-Klein, Dirac quantized classical Hamiltonian H for harmonic oscillator by replacing physical quantities in it with corresponding self-adjoint operators as

$$H_{osc} = p^2/2m + m\omega^2 q^2/2m$$

where p and q are operators that satisfy the commutation $[p, q] = i\hbar$. Though the connection between this purely "formal" quantization and de Broglie's (or Schrödinger-Heisenberg) quantization is not understood as well as it should be, this easy going formal quantization took over and became standard in contemporary quantum field theory.

Notwithstanding, with p and q, we define the non-commuting operators

$$a = (m\omega p + ip)/\sqrt{2\hbar m\omega} \qquad a^+ = (m\omega p - ip)/\sqrt{2\hbar m\omega}.$$

It is clear that $[a, a^+] = 1$. Now we have

$$H_{osc} = (1/2)\hbar\omega(a^+a + aa^+) = \hbar\omega(a^+a + 1/2).$$

Define N as $N = a^+a$. It follows that:

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1. Eigenvalues of N are $n = 0, 1, 2, ...$

2. If $|n\rangle$ is normalized then so are $|n \pm 1\rangle$ defined as

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad a^+|n\rangle = \sqrt{n+1}|n+1\rangle.$$

If $|0\rangle$ is normalized, the normalized eigenvectors of N are $|n\rangle = ((a^+)^n/\sqrt{n!})|0\rangle$, where $n = 0, 1, 2, ...$ These are also eigenvectors of H_{osc} with eigenvalues $E_n = \hbar\omega(n + 1/2)$, for $n = 0, 1, 2, ...$The operators a and a^+ are called *annihilation operator* and *creation operator*, respectively. This is because $|n\rangle$ represents a quantum state with n quanta.

In summary, the quantized Hamiltonian for harmonic oscillator can be expressed using creation operator a and annihilation operator a^+ as

$$H_{osc} = (1/2)\hbar\omega(a^+a + aa^+).$$

What is not clear here is the relation between quantum particles (quanta) derived from H_{osc} and the original particle which was described as H. Traditionally, the Hamiltonian represents a classical single particle system. Dirac produced many particles from it. Moreover, many particle systems are non-linear and have no analytic solution. This problem was pointed out by Prof. Lehto. All of this means that *there is no clear ontological meaning to the quanta Dirac created from H_{osc}. This means that there is no ontology behind $|n\rangle$.*

Remark 1 *Moreover, as mentioned above, there is no clear connection between Schrödinger's observables (self-adjoint operators) and Dirac's observables, except that both of them suffer from the deficiency of the "uncertainty problem".*

2 Quantization of electromagnetic field: Dirac's aether theory

Planck quantized energy of electromagnetic waves to deal with the black-body radiation problem. Dirac went on to quantize the electromagnetic field which is supposed to be the medium for electromagnetic waves of Maxwell. This is called the *"second quantization"*.

2.1 Scalar and vector potential

Through Fourier expansion of the electromagnetic field represented by the vector potential field, Dirac induced photons as harmonic oscillators in the space together with the creation and annihilation operator.

According to the classical theory of electromagnetism, there are a scalar potential ϕ and a vector potential $\mathbf{A}$ such that the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ of Maxwell can be obtained as

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\phi, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}.$$

If there is no source of the field, we choose a gauge (Coulomb gauge) such that $\phi = 0, \boldsymbol{\nabla} \cdot \mathbf{A} = 0$. From these equations we can derive the Maxwell equation of electromagnetic fields.

From these, we have the following "wave equation of vector potential".

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \qquad (I)$$

This means that vector potential $\mathbf{A}$ for charge-free space is a wave. But as $\mathbf{E}$ and $\mathbf{B}$ are modality, $\mathbf{A}$ is not physical reality but modality. So, $\mathbf{A}$ is not a physical wave but *a modal wave*. Let us call this wave "*(vector) potential wave*". First question is that there are infinitely many vector potentials $\mathbf{A}$ that satisfy this wave equation. Which one are we going to discuss? *On what grounds do we make this decision*?

2.2 Quantization of electromagnetic field

Following Dirac, we make a Fourier expansion of the electromagnetic field in a large cube of volume $\Omega = L^3$ and take the Fourier coefficients as the field variables. We choose the boundary conditions to be periodic on the walls of the cube. This is

$$\mathbf{A}(L, y, z, t) = \mathbf{A}(0, y, z, t), \quad \mathbf{A}(x, L, z, t) = \mathbf{A}(x, 0, z, t), \quad \mathbf{A}(x, y, L, t) = \mathbf{A}(x, y, 0, t).$$

The Fourier series of $\mathbf{A}$ is given by

$$\mathbf{A}(\mathbf{x},t) = \sum_{\substack{\mathbf{k} \\ k_z>0}} \sum_{\sigma=1,2} \sqrt{2\pi\hbar c^2/\Omega\omega_k}\,\mathbf{u}_{\mathbf{k}\sigma}(a_{\mathbf{k}\sigma}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}\sigma}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}) \qquad (II)$$

where $\mathbf{k}$ is a wave vector, $\omega_k = kc$ and $k = \langle \mathbf{k} \cdot \mathbf{k} \rangle$. The factor $\sqrt{2\pi\hbar c^2/\Omega\omega_k}$ is a normalization factor. $\mathbf{u}_{\mathbf{k}\sigma}$, $\sigma = 1, 2$ are two orthogonal unit vectors. Due to the second condition of the Coulomb gauge, they must be orthogonal to the wave vector $\mathbf{k}$ which has the components

$$2\pi(n_x, n_y, n_z)/L$$

where n_i are integers.

From (II) to (I), with

$$a_{\mathbf{k}\sigma}(0) = \begin{cases} a^{(1)}_{\mathbf{k}\sigma}(0) & \text{if } k_z > 0 \\ a^{(2)}_{-\mathbf{k}\sigma}(0) & \text{otherwise} \end{cases}$$

where

$$a_{\mathbf{k}\sigma}(t)e^{i\mathbf{k}\cdot\mathbf{x}} = a_{\mathbf{k}\sigma}(0)e^{-i\mathbf{k}\cdot\mathbf{x}}$$

we have

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mathbf{k},\boldsymbol{\sigma}} \sqrt{2\pi\hbar c^2/\Omega\omega_k}\,\mathbf{u}_{\mathbf{k}\sigma}[a_{\mathbf{k}\sigma}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + a^{(1)^*}_{\mathbf{k}\sigma}(t)e^{-i\mathbf{k}\cdot\mathbf{x}})]$$

This leads to

$$da_{\mathbf{k}\sigma}(t)/dt = -i\omega_t a_{\mathbf{k}\sigma}$$

This equation for all wave vectors $\mathbf{k}$ and $\sigma = 1, 2$ can be considered as "*the equation of motions of the electromagnetic field*".

Now, the energy in the electromagnetic field (radiation Hamiltonian) is

$$H_{rad} = \frac{1}{8\pi}\int_{\Omega} d^3\mathbf{x}(E^2+B^2) = \int_{\Omega} d^3\mathbf{x}\left(\frac{1}{c^2}\left|\frac{\partial \mathbf{A}}{\partial t}\right|^2 + |\nabla \times \mathbf{A}|^2\right) = \frac{1}{2}\sum_{\mathbf{k},\sigma}\hbar\omega_k(a_{\mathbf{k}\sigma}a^*_{\mathbf{k}\sigma}+a^*_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}).$$

With this, we can consider the electromagnetic field to be an infinite collection of harmonic oscillators. Now we have

$$H_{rad} = \sum_{\mathbf{k},\sigma}\hbar\omega_k(\frac{1}{2} + a^*_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma}) \qquad (III)$$

and $\frac{1}{2}\hbar\omega_k$ is the *zero-point energy* of an oscillator. Then the zero-point energy of the radiation field

$$\sum_{\mathbf{k},\sigma}\frac{1}{2}\hbar\omega_k$$

is *infinite* as there are *infinitely* many oscillators. *As there are continuumly many wave vectors* $\mathbf{k}$, *there are continuumly many photons in the empty space.* This does not agree with physical ontology of particles.

As discussed, this problem is directly connected to the question of how many photons are there in the space? Photons are "supposed to be" physical particles. The problem here is that *if photons are to create continuum then photons cannot be physical particles.* A collection of ontological particles cannot form continuum. Planck-Einstein's quantization of light waves shares the same problem. *As there are continuumly many wave lengths for electromagnetic waves there must be continuumly many photons of Planck-Einstein, which is not possible.*

Remark 2 *Despite the indifference of quantum physicists, this makes the photon concept of Planck-Einstein invalid. The mathematics they used violates the ontology. To begin with, as this theory is invalid, what is the point of adding Dirac's quantization of electromagnetic field into this theory.*

Here, Dirac carried out the quantization of (local) electromagnetic field expressed by the vector potential $\mathbf{A}$. This is to produce *"quanta of electromagnetic fields"* as harmonic oscillators and the total energy of such electromagnetic field as the summation (integration to be precise) of the energy of such harmonic oscillators. This result suffers from serious "category errors". Electromagnetic field is not a physical reality. It is a counterfactual modality. So, the produced quanta of harmonic oscillators must not be considered as physical reality. *They are just a fancy mathematical representation of this metaphysical world of electromagnetic fields which does not exist in physical reality.* How can the concept of the spatial distribution of electric force per unit charge be a physical reality. In Dirac's eccentric world, where symbolic calculation is the only truth, "objects" defined from counterfactual modality through formal symbol pushing produce physical reality of "photons" whose connection to Planck-Einstein's photons is not presented at all.

Moreover, there is yet another good reason to question Dirac's claim that photons are in essence the components wave functions which appear in the Fourier expansion of the electromagnetic field expressed by the vector potential **A**. This means that Dirac's photon is an "infinite object" and this does not go quite well with the assumption that photons are "the most basic elementary particle".

The connection between Dirac's photons and Einstein-Planck's photons is not as clear as it should be. Dirac's photons are quantization of electromagnetic fields and Planck-Einstein's photons are quantization in terms of energy of electromagnetic waves of Maxwell. Certainly, as waves that "travel through" the counterfactual modality of electromagnetic field, electromagnetic waves are also counterfactual modality, "not reality".

Furthermore, contrary to the belief of Dirac, they are not the same things. What we can see in common here is the issue of *"mathematically producing physical particles through quantizing non-physical entities such as electromagnetic fields and electromagnetic waves".* Moreover, Planck-Einstein photons are invalid as they lead to theoretical contradictions and the empirical contradiction of violating the uncertainty principle as discussed above.

Recent study shows that electromagnetic fields should be represented as the system of monochromatic operators instead to prevent the problem of black-body radiation. Though this by itself will not provide a solution to the problem of particle-wave duality, which is a very deep mathematical and philosophical problem, it at least seems to "explain" the black-body radiation. After all, choosing the harmonic oscillator or the monochromatic oscillator for photon's mathematical representation has no ontological reasoning. So, this is a good example of how quantum theory violates the empiricism. *It is tragic that physicists who claim that mathematics is just a language abuse mathematics to deal with inconvenient empirical issues like this.*

The most fundamental issue is that *it is not the case that we empirically detected particles called photons and we found a mathematical representation of them.* Photons here are nothing but the creation of this rather elementary mathematical construction of Fourier expansions and Dirac decided that they are physical particles called "photons" without considering their connection to yet another kind of photons presented by Planck who refused to consider his photons to be particles.

2.3 Annihilation operator and creation operator

Again, following the steps of Gordon-Klein, Dirac further "quantized" the above presented quantization of the classical radiative field by replacing the classical quantities $a_{\mathbf{k}\sigma}$ and $a^*_{\mathbf{k}\sigma}$ with self-adjoint operators. We may write $a_\sigma(\mathbf{k})$ and $a^*_\sigma(\mathbf{k})$ for $a_{\mathbf{k}\sigma}$ and $a^*_{\mathbf{k}\sigma}$. We just consider $a_\sigma(\mathbf{k})$ and $a^*_\sigma(\mathbf{k})$ quantum operators.

We assume that the operators referring to different oscillators commute, that is $[a_\sigma(\mathbf{k}), a^*_\sigma(\mathbf{k}')] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma\sigma'}$.

The operator $N_\sigma(\mathbf{k}) = a^*_\sigma(\mathbf{k}), a_\sigma(\mathbf{k})$ then has eigenvalues $n_\sigma(\mathbf{k})$, $n = 0, 1, 2, ...$ and eigenvectors defined as

$$a_\sigma(\mathbf{k})|n_\sigma(\mathbf{k})\rangle = \sqrt{n_\sigma(\mathbf{k})}|n_\sigma(\mathbf{k}) - 1\rangle, \quad a^*_\sigma(\mathbf{k})|n_\sigma(\mathbf{k})\rangle = \sqrt{n_\sigma(\mathbf{k})}|n_\sigma(\mathbf{k}) + 1\rangle.$$

Indeed,

$$|n_\sigma(\mathbf{k})\rangle = [[a^*_\sigma(\mathbf{k})]^{n_\sigma(\mathbf{k})}/\sqrt{n_\sigma(\mathbf{k})!}]|0\rangle.$$

The eigenvector of the radiation Hamiltonian given as equation (III) is a tensor product of such states, i.e.,

$$|\cdots n_\sigma(\mathbf{k})\cdots\rangle = \prod_{\mathbf{k},\sigma} |n_\sigma(\mathbf{k})\rangle \qquad (IV)$$

with the energy eigenvalues

$$E = \sum_{\mathbf{k},\sigma} \hbar\omega_k(n_\sigma(\mathbf{k}) + \frac{1}{2}). \qquad (V)$$

The interpretation of these equations is a straight forward generalization from one harmonic oscillator to a superposition of independent oscillators, one for each radiation mode $(\mathbf{k}, \sigma)$. $a_\sigma(\mathbf{k})$ operating on the state (IV) will reduce the energy while leaving the occupational numbers unchanged. Indeed, we have

$$|a_\sigma(\mathbf{k})|\cdots n_\sigma(\mathbf{k})\cdots\rangle = \sqrt{n_\sigma(\mathbf{k})}|\cdots n_\sigma(\mathbf{k}) - 1\cdots\rangle \qquad (VI).$$

Correspondingly, the energy (V) is reduced by $\hbar\omega_k = hc|\mathbf{k}|$.

We interpret $a_\sigma(\mathbf{k})$ as an "annihilation operator" which annihilates one photon in the model $(\mathbf{k}, \sigma)$, i.e. with momentum $\hbar\mathbf{k}$, energy $\hbar\omega_k$ and linear polarization vector $\mathbf{u}_{\mathbf{k}\sigma}$. Similarly, $a^*_\sigma(\mathbf{k})$ is interpreted as a "creation operator" of such a photon. We have

$$|a_\sigma(\mathbf{k})|\cdots n_\sigma(\mathbf{k})\cdots\rangle = \sqrt{n_\sigma(\mathbf{k})}|\cdots n_\sigma(\mathbf{k}) + 1\cdots\rangle \qquad (VII).$$

The state of the lowest energy of the radiation field is the "vacuum state" $|0\rangle$ in which all occupational numbers $n_\sigma(\mathbf{k})$ are zero. In lieu of (V), this state has energy

$$\frac{1}{2}\sum_{\mathbf{k},\sigma} \hbar\omega_k.$$

Quantum field theory works only for the systems for which the zero-point energy of the radiative field cancels out. *For "many cases", this infinite energy of vacuum cancels out when physically meaningful quantities are calculated.* So, we "assume"

$$H_{rad} = \sum_{\mathbf{k},\sigma} \hbar\omega_k a^*_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}.$$

Remark 3 *Is this diverging zero-point energy as a part of the formal mathematical representation of the classical electromagnetic field not an indication of the unacceptability of Dirac's theory of quantization of electromagnetic field? This problem comes from very serious and deep issues we have in theoretical physics that transcend the opportunistic conventionalism which comes from the empirical tradition of physics. It is hard to imagine that a serious mathematical and conceptual development we are engaged in here could lead to this kind of opportunistic conclusion.*

The eigenvalues of this operator are

$$E = \sum_{\mathbf{k},\sigma} \hbar\omega_k n_\sigma(\mathbf{k}).$$

The momentum operator is

$$\mathbf{P} = \sum_{\mathbf{k},\sigma} \hbar\mathbf{k}(a^*_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}) = \sum_{\mathbf{k},\sigma} \hbar\mathbf{k}(N_\sigma(\mathbf{k}))$$

whose eigenvalues are

$$\sum_{\mathbf{k},\sigma} \hbar\mathbf{k}(n_\sigma(\mathbf{k})).$$

In conclusion, the following picture of the electromagnetic field emerges: It consists of photons each of which has energy $\hbar\omega_k$ and momentum $\hbar k : n_{\mathbf{k}\sigma}$ is the number of photons with momentum $\hbar\mathbf{k}$.The polarization is given by the vector $\mathbf{u}_{\mathbf{k}\sigma}$. The annihilation operator $a_{\mathbf{k}\sigma}$ decreases the number of photons with the momentum $\hbar\mathbf{k}$ by one and the creation operator $a^*_{\mathbf{k}\sigma}$ increases the number of photons with the momentum $\hbar\mathbf{k}$ by one.

3 Dirac's quantization of Schrödinger's wave equation (the second quantization)

Without knowing any of these fatal issues with his quantization of electromagnetic fields, Dirac went on to apply the same idea to the Schrödinger wave equations. If this was Dirac's final answer to the frustrating problem of the failure to make Schrödinger's wave equation relativistic is not quite clear.

Consider Schrödinger's equation

$$-\frac{\hbar}{i}\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V(\mathbf{x})\Psi$$

for a particle in a potential $V(\mathbf{x})$. Let Ψ_n and E_n be the eigenvectors and eigenvalues of the operator $-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})$. This is to say

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right)\Psi_n = E_n\Psi_n.$$

The "Fourier expansion" of the wave function is

$$\Psi(\mathbf{x}, t) = \sum_n b_n(t)\Psi_n(\mathbf{x}).$$

Substituting this to the Schrödinger equation yields

$$\frac{d}{dt}b_n = -\frac{1}{h}E_n b_n.$$

The expected value of the energy is

$$H = \int d^3x \Psi^*(\mathbf{x}, t)\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right]\Psi(\mathbf{x}, t).$$

Putting all of these equations together and considering the orthogonality of Ψ_n we have

$$H = \sum_n E_n b_n^* b_n.$$

This is the Hamiltonian for a collection of harmonic oscillators with frequencies $E_n/\hbar$. If we consider b_n as an operator then b_n^* can be considered as the adjoint of b_n, in symbols, b_n^+. Under the commuting relations

$$[b_n, b_{n'}] = [b_n^+, b_{n'}^+] = 0, \qquad [b_n, b_n^+] = 0$$

from Heisenberg's equation

$$-\frac{\hbar}{i}\frac{d}{dt}b_n = [b_n, H]$$

we can derive Fourier version of the Schrödinger equation as planned.

In this way a Schrödinger's wave equation which is obtained from Hamiltonian is represented by an infinite system of oscillating particles, as Dirac planned.

The operators $b_n^+ b_n$ have the eigenvalues $N_n = 0, 1, 2, ...$, indicating that any natural number of particles may occupy the eigenstate Ψ_n. Then, the eigenvalue of H is

$$E = \sum_n E_n N_n.$$

This theory obeys the Bose-Einstein statistics and these particles are called *bosons*.

This theory excludes particles that obey the Fermi-Dirac statistics. These particles are called *fermions*. A minor change of the theory above will derive a theory of fermions. We keep the Hamiltonians as

$$H = \sum_n E_n b_n^* b_n.$$

We expect the Heisenberg equation of motion to yield

$$\frac{d}{dt}b_n = -\frac{1}{\hbar}E_n b_n.$$

The only change involved is the commuting relations

$$[b_n, b_{n'}] = [b_n^+, b_{n'}^+] = 0, \qquad [b_n, b_n^+] = 0$$

to the commuting relations

$$[b_n, b_{n'}]_+ = [b_n^+, b_{n'}^+]_+ = 0, \qquad [b_n, b_n^+]_+ = \delta_{n,n'}$$

where $[A, B]_+ = AB + BA$. Now we have

$$-\frac{\hbar}{i}\frac{d}{dt}b_n = [b_n, H] = \sum_m E_m\{b_n b_m^+ b_m - b_m^+ b_m b_n\} = \sum_m E_m \delta_{nm} b_m = E_n b_n.$$

So, we have obtained Heisenberg's equation of motion. Note that

$$(b_n^+ b_n) b_n^+ b_n = b_n^+(1 - b_n^+ b_n) b_n = b_n^+ b_n - b_n^+ b_n b_n^+ b_n = b_n^+ b_n.$$

If λ is an eigenvalue of $b_n^+ b_n$ then

$$b_n^+ b_n |\lambda\rangle = \lambda|\lambda\rangle \qquad b_n^+ b_n b_n^+ |\lambda\rangle = \lambda^2 |\lambda\rangle = \lambda|\lambda\rangle.$$

Thus $\lambda^2 = \lambda$. This is to say $\lambda = 1$ or $\lambda = 0$. This means that *at most one particle can occupy the eigenstate* Ψ_n. We may write $|n\rangle$ to denote this eigenstate. This theory obeys the Fermi-Dirac statistics.

To express all of this on λ, we may write $b_n^+ b_n |N_n\rangle = N_n |N_n\rangle$ where $N_n = 0, 1$. Now we have

$$b_n^+ b_n b_n^+ |N_n\rangle = b_n^+(1 - b_n b_n^+)|N_n\rangle = (1 - N_n) b_n^+ |N_n\rangle.$$

This implies that $b_n^+|N_n\rangle$ is an eigenvector of $b_n^+ b_n$ with the eigenvalue $1 - N_n$. It can differ from $|1 - N_n\rangle$ only by a constant. We write

$$b_n^+ |N_n\rangle = C_n |1 - N_n\rangle.$$

The constant C_n can be evaluated by taking the inner product of $b_n^+|N_n\rangle$ with itself.

$$\langle b_n^+|N_n\rangle, b_n^+|N_n\rangle\rangle = (1 - N_n) = C_n^* C_n.$$

Thus we have

$$C_n = \theta_n \sqrt{1 - N_n}$$

where θ_n is a phase factor of modulus unity. This leads to

$$b_n^+|N_n\rangle = \theta_n \sqrt{1 - N_n}|1 - N_n\rangle \qquad b_n|N_n\rangle = \theta_n \sqrt{N_n}|1 - N_n\rangle.$$

In summary we have

1. For bosons:

$$b_n|\cdots, N_n, \cdots\rangle = \sqrt{N_n}|\cdots, N_n - 1, \cdots\rangle b_n^+|\cdots, N_n, \cdots\rangle = \sqrt{N_n}|\cdots, N_n + 1, \cdots\rangle$$

2. For fermions:

$$b_n|\cdots,N_n,\cdots\rangle = \theta_n\sqrt{N_n}|\cdots,1{-}N_n,\cdots\rangle \qquad b_n^+|\cdots,N_n,\cdots\rangle = \theta_n\sqrt{1-N_n}|\cdots,1{-}N_n,\cdots\rangle$$

where $N_n = 0, 1$.
In both cases, b_n is a annihilation operator and b_n^+ is a creation operator.

Remark 4 *One more question remains to be answered. Why Dirac started with a second quantization of Schrödinger's wave equations? Why did he not start directly with Hamiltonians? It was because Hamiltonians are just classical equations of energies. Dirac wanted to quantize energy fields in general as he did to electromagnetic fields so that the same theoretical framework would apply to energies in general. He thought that waves are fields. There is a vicious circularity in his reasoning. Waves assume fields but not vice versa.*

4 Interactions of quantum particles

We can add the Hamiltonians for several free particle fields and introduce appropriate interaction terms to study interacting particle fields. The most common such interaction is that of photons with charged particles. We use the theory of second quantization to represent a charged particle field by the following Hamiltonian:

$$\int d^3x\Psi^+(\mathbf{x},t)\left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\Psi(\mathbf{x},t).$$

The quantized electromagnetic field is represented by the following radiation (photon) Hamiltonian:

$$\int d^3x\frac{1}{8\pi}(E^2+B^2).$$

The interaction of these two fields will be obtained by adding these two Hamiltonians and prescribing the following replacement:

$$\frac{\hbar}{i}\nabla \Longrightarrow \frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}(\mathbf{x}).$$

This leads to

$$H = \int d^3\mathbf{x}\Psi^+(\mathbf{x},t)\left[-\frac{\hbar^2}{2m}\left|\frac{\hbar}{i}\nabla - \frac{e}{c}\mathbf{A}(\mathbf{x})\right|^2 + V\right]\Psi(\mathbf{x},t) + \int d^3x\frac{1}{8\pi}(E^2+B^2) = H_P + H_{rad} + H_I$$

where

$$\int d^3x\Psi^+(\mathbf{x},t)\left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\Psi(\mathbf{x},t) = \sum_n E_n b_n^+ b_n$$

is the particle Hamiltonian,

$$H_{rad} = \int d^3\mathbf{x}\frac{1}{8\pi}(E^2+B^2) = \sum_{k,\sigma}\hbar\omega_k a^+_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}$$

is the Hamiltonian for the radiation field, and

$$H_I = \int d^3x \Psi^+(\mathbf{x},t)\left[-\frac{\hbar^2}{imc}\mathbf{A}\cdot\nabla^2 + \frac{e^2}{2mc^2}A\right]\Psi(\mathbf{x},t)$$

is the interaction Hamiltonian. We can divide H_I into a part H' proportional to A and a part A' proportional to A^2 such that $H_I = H' + H$". Expanding A and Ψ in terms of $a_{\mathbf{k},\sigma}$ and b_n gives

$$H' = \sum_{\mathbf{k},\sigma}\sum_{n}\sum_{n'}\left[M(\mathbf{k},\sigma,n,n')b_n^+ b_{n'} a_{\mathbf{k},\sigma} + M(-\mathbf{k},\sigma,n,n')b_n^+ b_{n'} a_{\mathbf{k},\sigma}^+\right]$$

and

$$\begin{aligned} H\text{"} \quad = \quad & \sum_{\mathbf{k}_1,\sigma_1}\sum_{\mathbf{k}_2,\sigma_2}\sum_{n}\sum_{n'} M(\mathbf{k}_1,\sigma,\mathbf{k}_2,\sigma_2,n,n')a_{\mathbf{k}_1,\sigma_1}a_{\mathbf{k}_2,\sigma_2} + M(\mathbf{k}_1,\sigma_1,-\mathbf{k}_2,\sigma_2,n,n')a_{\mathbf{k}_1,\sigma_1}a_{\mathbf{k}_2,\sigma_2}^+ \\ & + M(-\mathbf{k}_1,\sigma_1,\mathbf{k}_2,\sigma_2,n,n')a_{\mathbf{k}_1,\sigma_1}^+ a_{\mathbf{k}_2,\sigma_2} + M(-\mathbf{k}_1,\sigma_1,-\mathbf{k}_2,\sigma_2,n,n')a_{\mathbf{k}_1,\sigma_1}^+ a_{\mathbf{k}_2,\sigma_2}^+ \end{aligned}$$

where

$$M(\mathbf{k},\sigma,n,n') = \sqrt{\frac{2\pi\hbar c^2}{\Omega\omega_k}\int d^3\mathbf{x}\Psi_n^*\left[-\frac{e\hbar}{imc}e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{u}_{k,\sigma}\cdot\nabla\right]}\Psi_{n'}$$

and

$$\begin{aligned} & M(\mathbf{k}_1,\sigma,\mathbf{k}_2,\sigma_2,n,n') \\ = \quad & \sqrt{\frac{2\pi\hbar c^2}{\Omega\omega_k}}\sqrt{\frac{1}{\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2}}}\int d^3\mathbf{x}\Psi_n^*\left[-\frac{e\hbar}{2mc^2}e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{u}_{\mathbf{k}_1,\sigma_1}\cdot\mathbf{u}_{\mathbf{k}_2,\sigma_2}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}\right]\Psi_n. \end{aligned}$$

The part of the Hamiltonian $H_p + H_{rad}$ can be considered the unperturbated part with eigenvectors $|\cdots N_n \cdots\rangle_p |\cdots n_{\mathbf{k},\sigma}\cdots\rangle_{rad}$ and eigenvalues $\sum_n E_n N_n + \sum_{\mathbf{k},\sigma}\hbar\omega_{\mathbf{k}} n_{\mathbf{k}\sigma}$.

The interaction Hamiltonian H_I induces transitions between these states as follows:

1. the term $b_n^+ b_{n'} a_{\mathbf{k},\sigma}$ in $H^{'}$: (1) annihilates a photon of momentum $\hbar\mathbf{k}$ and polarization σ, (2) annihilates a particle in state $|n'\rangle$, (3) creates a particle in state $|n\rangle$.

2. the term $b_n^+ b_{n'} a_{\mathbf{k},\sigma}$ in $H^{'}$: (1) creates a particle in state $|n\rangle$, (2) annihilates a particle in state $|n'\rangle$, (3) annihilates a photon of momentum $\hbar\mathbf{k}$ and polarization σ.

3. the term $b_n^+ b_{n'} a_{\mathbf{k},\sigma}$ in $H^{''}$: (1) creates a particle in state $|n'\rangle$, (2) annihilates a particle in state $|n\rangle$, (3) annihilates a photon of momentum $\hbar\mathbf{k}_1$ and polarization σ_1, (4) annihilates a photon of momentum $\hbar\mathbf{k}_2$ and polarization σ_2.

4. the term $b_n^+ b_{n'} M a_{\mathbf{k}_1,\sigma_1} a^+_{\mathbf{k}_2,\sigma_2}$ in H'': (1) creates a particle in state $|n\rangle$, (2) annihilates a particle in state $|n'\rangle$, (3) annihilates a photon of momentum $\hbar\mathbf{k}_1$ and polarization σ_1, (4) create a photon of momentum $\hbar\mathbf{k}_2$ and polarization σ_2.

5. the term $b_n^+ b_{n'} M a^+_{\mathbf{k}_1,\sigma_1} a_{\mathbf{k}_2,\sigma_2}$ in H'': (1) creates a particle in state $|n\rangle$, (2) annihilates a particle in state $|n'\rangle$, (3) create a photon of momentum $\hbar\mathbf{k}_1$ and polarization σ_1, (4) annihilates a photon of momentum $\hbar\mathbf{k}_2$ and polarization σ_2.

6. the term $b_n^+ b_{n'} M a^+_{\mathbf{k}_1,\sigma_1} a_{\mathbf{k}_2,\sigma_2}$ in H'': (1) creates a particle in state $|n\rangle$,(2) annihilates a particle in state $|n'\rangle$, (3) create a photon of momentum $\hbar\mathbf{k}_1$ and polarization σ_1, (4) create a photon of momentum $\hbar\mathbf{k}_2$ and polarization σ_2.

5 Lost in "Mathematics"

As we have seen, Dirac's quantum field theory is lost in mathematics or in a lack of it. Even tough it is very hard to capture microscopic particles as we cannot put them on our hands and examine, it is not an excuse to just jump into careless mathematization and push the theory to its limit without knowing that something went very wrong. Things that make sense mathematically may well not make much sense physically. The connection between mathematics and physics has to be re-examined. We have come a long way since Newton presented a good mathematization of ontology.

The continuum and discrete structure are entirely different things. Being countable is not a necessarily condition for being infinite. According to set theory, there is an infinitely high hierarchy of infinite sets. Countably infinite sets are at the bottom of this hierarchy. The so-called calculus, which is the most fundamental mathematics for physics, is based upon uncountable continuum mathematical structures. The reason why we need this uncountable mathematics for the theory of point masses is that the motion itself is a continuum mathematical structure. A particle will not jump from one point to the other. It moves continuously. This means we cannot use this mathematics to study discrete structure. This is where Dirac's work got derailed. He identified continuum and discrete.

Feynman's quantum electrodynamics is more radical in getting lost in mathematics. That a particle can move backward in time is not physics, though it can be introduced mathematically. Regarding this in the context of GTR, Gödel pointed out that one can build an interpretation of GTR in which time evolves backward. Einstein did not respond. As we discussed in the first part of our presentation, Minkowski's 4D spacetime theory with the Lorentz transformation is yet another example of getting lost in mathematics. Mathematically, it is hard to guess what it is about at all. Nobody, including Minkowski, seems to really understand it.

In mathematics, we do have equally serious crisis. Mathematicians run out of things to do. It is tragic that the most important work of mathematics in the last 100 years is the solution to Fermat's last theorem. Gauss, a greatest mathematician in history, said clearly that this problem is not worth wasting time on. He was correct. Pretty much at the same time, the Russian mathematician Perelman rejected the offer of Fields award for solving the Poincare conjecture. He said that this problem is not important.

Diverging from theoretical physics, mathematicians lost inspirations and motivations and have been left with just picking fallen autumn leaves. Mathematics is dead and physics is a popular science now! It is time that physicists and mathematicians go back to the roots and start working together again.